EDUCATIONAL MONOGRAPH

prepared from

Controller Design for Nonlinear and Time-Varying Plants

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FOREWORD

This monograph was produced at Virginia Polytechnic Institute in a pilot program administered by Oklahoma State University under contract to the NASA Office of Technology Utilization. The program was organized to determine the feasibility of presenting the results of recent research in NASA Laboratories, and under NASA contract, in an educational format suitable as supplementary material in classwork at engineering colleges. The monograph may result from editing single technical reports or synthesizing several technical reports resulting from NASA's research efforts.

Following the preparation of the monographs, the program includes their evaluation as educational material in a number of universities throughout the country. The results of these individual evaluations in the classroom situation will be used to help determine if this procedure is a satisfactory way of speeding research results into engineering education.

ABSTRACT

This monograph discusses a technique to generate a control signal which forces the state of a nonlinear plant to be close to the state of a reference model. The method is suitable for a broad class of nonlinear plants. Special emphasis is placed on the time of response due to perturbations from equilibrium.

INSTRUCTOR'S GUILD FOR MONOGRAPHS

1. Educational level of the monograph - Senior or beginning graduate level.
2. Prerequisite course material - Some introductory modern control theory including the concepts of state-space modeling.
3. Estimated lecture time required - One hour.
4. Technical significance of the monograph - The material presented in this monograph illustrates the power and versatility of Liapunov's Second Method. Very little material is available on the Second Method applied to a design problem. The technique described herein is conceptually a very broad solution to the controller design problem. The main difficulty seems to be the cost of implementing the design criteria as compared with other types of solutions such as the "on-line computer" controller.
5. New concepts or unusual concepts illustrated - This technique represents an extension to ideas previously developed in reference 3.
6. How monograph can best be used -

   Following the presentation of material, the class may benefit from either

   (a) Synthesizing a controller for a plant containing a type of nonlinearity different than that of the example, or

   (b) Comparing the controller developed in the example with that developed by another technique.
7. Other literature of interest to this subject is listed in the bibliography.
8. Note to Instructor: All uncolored pages of the instructors monograph are in the copies intended for student use.

CONTROLLER DESIGN FOR NONLINEAR AND TIME-VARYING PLANTS

Introduction

Prior to the last decade the theory and techniques of control engineers were focused primarily on the design of controllers, or regulators, for systems (plants) which could be described by linear differential equations. As a result the controller's function was fulfilled only over a certain range of operation since all physical plants are nonlinear to some degree. In the sense that the plant is amenable to being represented by a linear model there is conceptually no difficulty in obtaining the dynamic performance of the system. However, to meet the demands of modern industrial automation as well as those of military and space systems, the limitation of a linear plant has been removed from many theoretical approaches, thus increasing the complexity of the controller design.

This monograph presents an approach to the controller design problem for the case where the type of nonlinearity of the plant is known beforehand. The technique is an application of Liapunov's Second Method, and although a great deal has been written concerning Liapunov's Second Method with respect to stability problems, there are notably few cases in which this powerful method has been applied to engineering design problems.

Liapunov's Second Method

The second or direct method of Liapunov is concerned with determining the stability of a differential system without having to solve the equations. The conceptual technique applies to systems that may be forced, nonlinear, time-varying and/or stochastic in nature. The method involves the construction (by any means possible) of a scalar-valued function of the state-variables of the system, which is analogous to the total energy of the system. If the time derivative of this "energy" function is negative for each value of the state-vector, then the associated equilibrium of the system is stable. (For a rigorous treatment of this material the reader is referred to Chapter 7 of Reference 1.)

Problem Statement: (See Figure 1)

Given a non-linear plant whose (state-variable) description is known (the allowable forms of the nonlinearity will be made explicit in the development which follows) and a model reference or idealized system which will be assumed to be both linear and passive.

To synthesize a controller which generates a signal to force the plant state toward the model state with minimum convergence time.

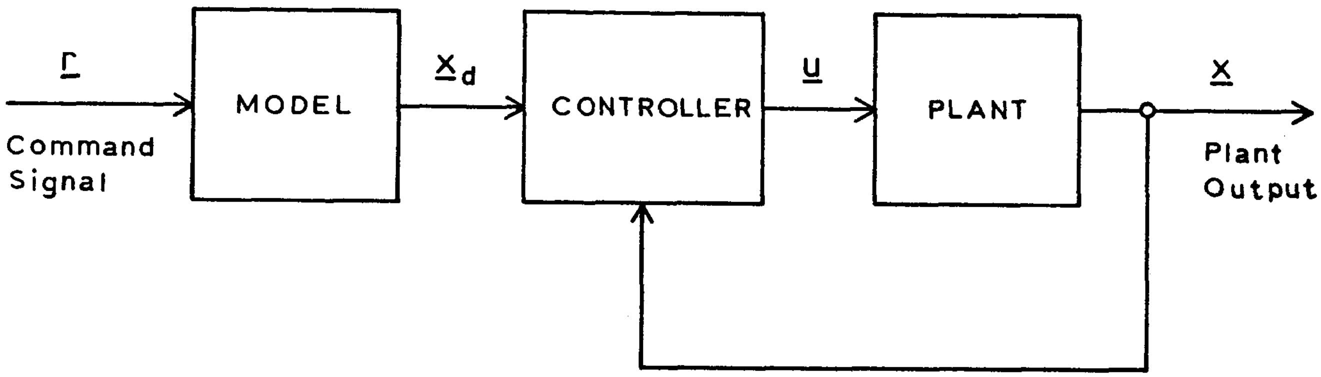

Fig. 1. System configuration.

Synthesis Technique:

By assumption the plant may be characterized by the state-variable (vector) differential equation

$$\dot{\underline{x}} = \underline{f}\,(\underline{x}, \underline{u}, \dot{\underline{u}}, \ldots\ldots, \underline{u}^{(n)}, t) \tag{1}$$

where

$\underline{u}$ is the control signal vector and

$\underline{x}$ is the state-vector of the plant

The presence of derivatives of $\underline{u}$ as arguments of $\underline{f}$ allows for the possibility of plant "zeros". Certain restrictions on $\underline{f}$ will be made as an outcome of the technique.

Also by assumption the ideal system behavior may be described by

$$\dot{\underline{x}}_d = A_o \underline{x}_d + B_o \underline{r} \tag{2}$$

where

$\underline{x}_d$ is the state vector of the reference model,

$\underline{r}$ is the command input vector and

A_o and B_o are (n x n) constant matrices.

From Equations (1) and (2) an error vector

$$\underline{e} = \underline{x}_d - \underline{x} \tag{3}$$

exists which must be reduced to zero by a suitable control signal $\underline{u}$. By subtracting Equation (1) from Equation (2):

$$\dot{\underline{e}} = \dot{\underline{x}}_d - \dot{\underline{x}} = A_o \underline{x}_d + B_o \underline{r} - \underline{f}$$

$$= A_o \underline{e} + A_o \underline{x} - \underline{f} (\underline{x}, \underline{u}, t) + B_o \underline{r} \tag{4}$$

Equation (4) is, then, a differential equation for the error vector and Liapunov's Second Method, which deals with the stability of differential systems, may be directed toward maintaining the "equilibrium", $\underline{e} = \underline{o}$, asymptotically stable in the large. In other words, it is desired to develop a practical method of generating the control vector $\underline{u}$ such that for any non-zero value of the error $\underline{e}$, the system will return to the equilibrium state, $\underline{e} = \underline{o}$.

A convenient starting point in the synthesis of the control signal $\underline{u}$ is the construction of a Liapunov function for the differential system in Equation (4).

Assume that the form of the Liapunov function is

$$V(\underline{e}) = \underline{e}^T P \underline{e} \tag{5}$$

where P is a symmetric matrix to be determined.

Taking the derivative of V with respect to time

$$\dot{V}(\underline{e}) = \dot{\underline{e}}^T P \underline{e} + \underline{e}^T P \dot{\underline{e}}$$

Substituting from equation (4):

$$\dot{V}(\underline{e}) = (\underline{e}^T A_o^T + \underline{x}^T A_o^T - \underline{f}^T + \underline{r}^T B_o^T) P \underline{e}$$

$$+ \underline{e}^T P (A_o \underline{e} + A_o \underline{x} - \underline{f} + B_o \underline{r})$$

$$\dot{V}(\underline{e}) = \underline{e}^T (A_o^T P + P A_o) \underline{e} + 2M \tag{6}$$

where

$$M = \underline{e}^T P A_o \underline{x} - \underline{f} (\underline{x}, \underline{u}, t) + B_o \underline{r}$$

Thus the equilibrium, $\underline{e} = \underline{o}$, will be asymptotically stable. Consequently V will be a Liapunov function for the error system described in Equation (4) if

1. $(A_o^T P + P A_o) = -Q$ is a negative definite matrix, and
2. The control signal $\underline{u}$ can be chosen to make the matrix M of Equation (6) non-positive.

Condition 1 can be met by a proper choice of P since A_o is the state matrix of a stable system by hypothesis. The second condition can be met provided that $\underline{f}$ satisfies certain requirements and its form and bounds are known. The restrictions on the type of nonlinearities to be allowed for the functions $\underline{f}$ are implicit in the method of choosing $\underline{u}$ such that M be a non-positive matrix. For the case where $\underline{f}$

is actually linear, the procedure illustrated below, requires only that the coefficient of the highest derivative of $\underline{u}$ be of one sign and non-vanishing.

Example:

To clarify the procedure of implementing condition 2 consider a simple second-order nonlinear time-varying plant described by the equation (see Figure 2)

$$\ddot{x} + a(t)(\dot{x})^2 + k\,b\,x = k\,u + k\,b\,r \tag{7}$$

Thus, the non-linearity is square-law damping. The state-space characterization of Equation (7) is found by letting $x_1 = x$ and $x_2 = \dot{x}$

$$\underline{\dot{x}} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{f}\,(\underline{x},\ \underline{u},\ t) = \begin{bmatrix} x_2 \\ -a(t)\,x_2^{\,2} - k\,b\,x_1 + k\,u + k\,b\,r \end{bmatrix} \tag{7'}$$

Assuming that the reference model equation is

$$\ddot{x}_d + a_o\,\dot{x}_d + k_o x_d = k_o r \tag{8}$$

The error equation becomes (subtracting Equation (7) from Equation (8)):

$$\ddot{e} + (a_o - a\,\dot{x})\dot{e} + (k_o - k\,b)e = (k_o - kb)r - ku \tag{9}$$

Letting $x_{d_1} = x_d$ and $x_{d_2} = \dot{x}_d$ for the state model of Equation (8):

$$\underline{\dot{x}}_d = \begin{bmatrix} 0 & 1 \\ -k_o & -a_o \end{bmatrix} \underline{x}_d + \begin{bmatrix} 0 \\ k_o \end{bmatrix} r \tag{8'}$$

By identifying matrices A_o and B_o from Equation (8'), a substitution may be made into Equation (6). Choosing Q diagonal:

$$\dot{V}(\underline{e}) = \dot{V}(e_1,\ e_2) = q_{11}e_1^{\,2} + q_{22}e_2^{\,2} + 2M \tag{10}$$

where

$$M = Z\begin{bmatrix}\xi_1 \xi_2\end{bmatrix} \begin{bmatrix} 0 & 1 \\ -k_o & -a_o \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 \\ k_o r \end{bmatrix} ,$$

$$\xi_1 = (e_1 p_{11} + e_2 p_{21}$$

and

$$\xi_2 = (e_1 p_{12} + e_2 p_{22})$$

so that

$$M = -2\ k\ \xi_2 \left[\frac{k_o - kb}{k} (x - r) + \frac{a_o}{k} \dot{x} - \frac{a(t)}{k} \dot{x}^2 + u \right]$$

Therefore, if a control signal u(t) can be generated to be instantaneously equal to

$$u = \left\{ \left[\frac{k - k_o}{k} \right] [r - x] + \left[\frac{a_o}{k} \right] [\dot{x}] + \dot{x}^2 \left[\frac{a}{k} \right]_{max} \right\} \text{ sign } \xi_2$$

then criterion 2 concerning Equation (6) will be satisfied. The block diagram for such a controller is incorporated in Figure 2.

Figure 3 illustrates the error vs. time of the system of Figure 2 where r(t) is the unit step. The error is seen to decay. In fact the error is never greater than 5% of the model reference.

<u>Convergence Time</u>:

An estimate of the convergence time is given by the parameter

$$\eta = \min \frac{-\dot{V}(\underline{e})}{V(\underline{e})}, \quad \underline{e} \neq \underline{0} \tag{12}$$

Therefore,

$$\dot{V}(\underline{e}) \leq - \eta V(\underline{e})$$

or

$$V(\underline{e}) \leq V_o\ e^{-\eta(t - t_o)} \tag{13}$$

where

$$V_o = V(\underline{e})\Big|_{t = t_o}$$

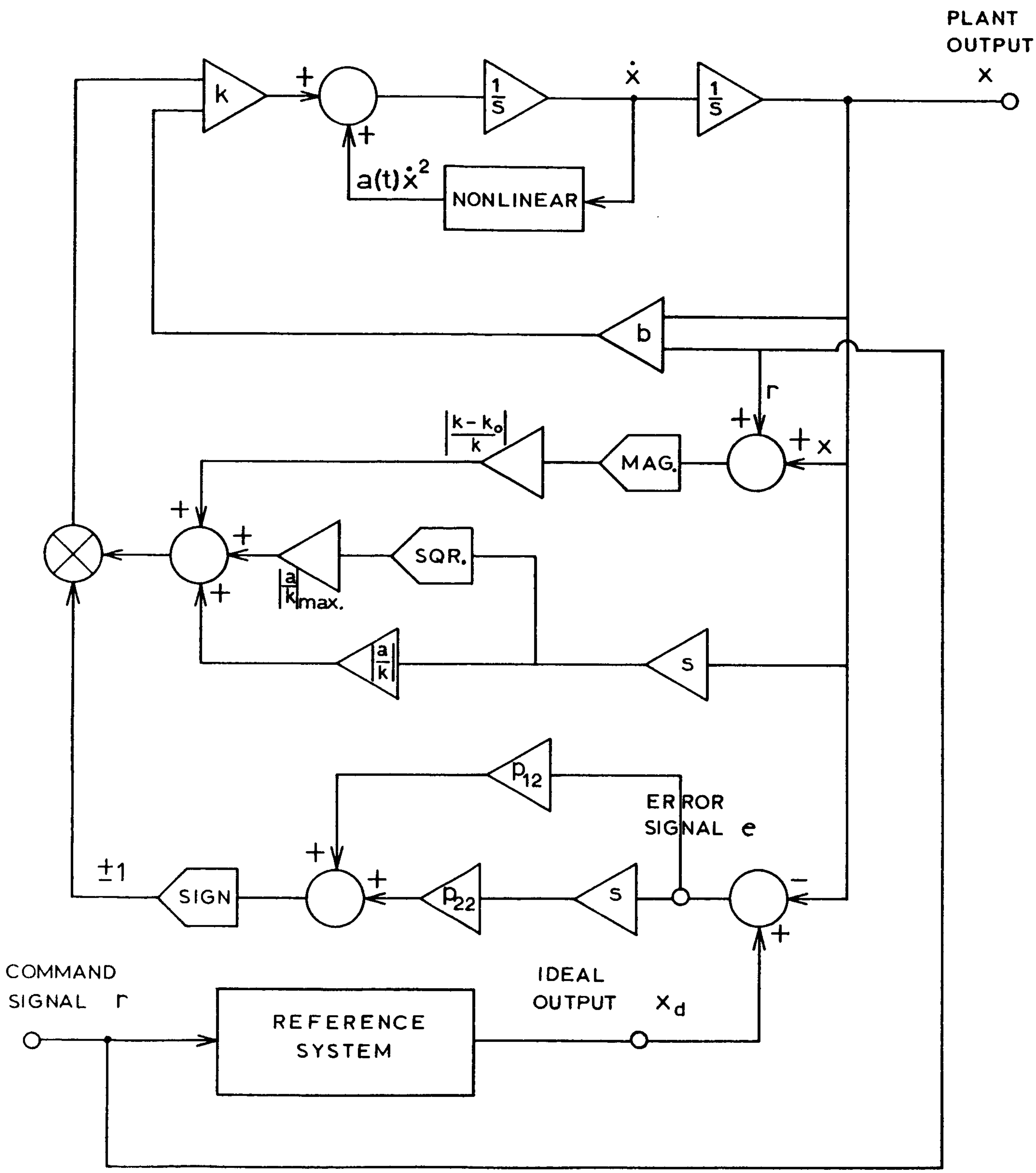

Figure 2: Total system block diagram for the Example. The plant (Eq. 7) is illustrated at top, the controller at bottom.

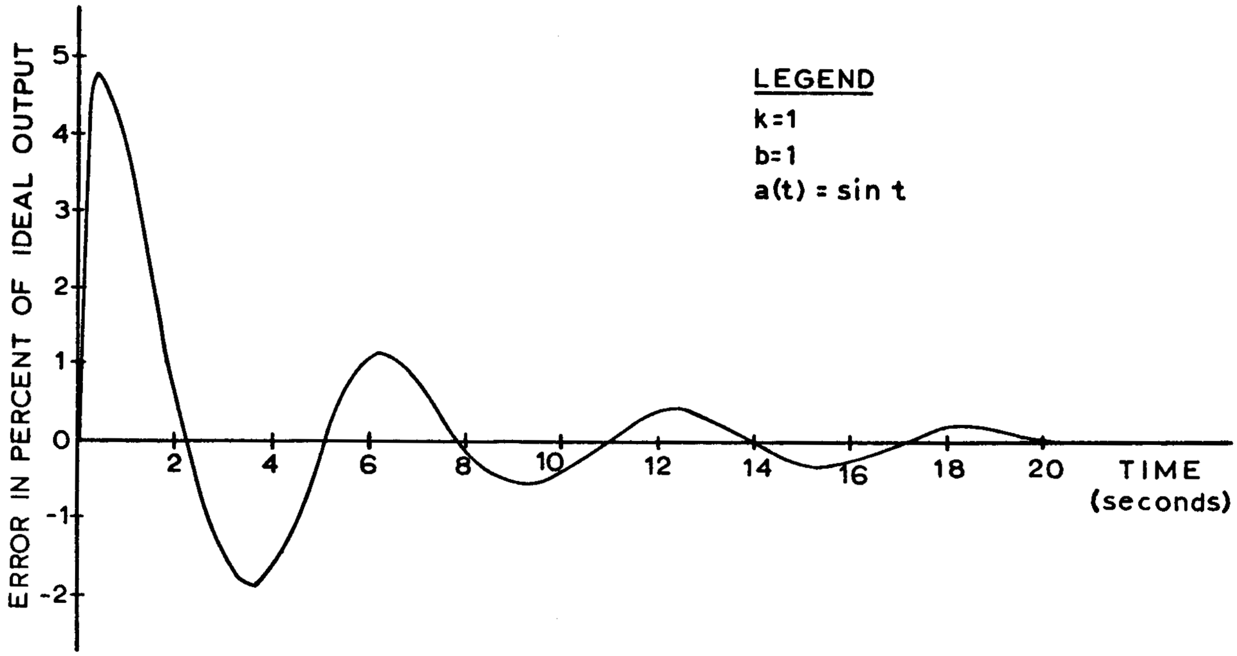

Fig. 3. Computer simulation giving the unit-step response of the example.

From Equation (13) the "energy" of the error system is decreased more rapidly if η is large. However, η is difficult to compute, but a related parameter

$$\eta_o = \frac{-\underline{e}^T(A_o^T P + P A_o)\,\underline{e}}{V(\underline{e})} = \frac{\underline{e}^T Q\,\underline{e}}{\underline{e}^T P\,\underline{e}} \tag{14}$$

is always less than or equal to η. Also η_o is much easier to maximize.

It is clear that the rate of convergence depends only on the matrix P since A_o is fixed by the design specifications.

A conservative estimate of the convergence time is then given in terms of the characteristic values of the matrices P and Q:

$$\eta \geq \eta_o \geq \frac{\text{The smallest eigenvalue of } Q}{\text{The largest eigenvalue of } P} = \alpha \tag{15}$$

In most cases this value of α as a lower bound to the convergence time η is too conservative to be of practical use.

Summary:

This monograph has presented a method which, although not original, is an unusual utilization of Liapunov's Second Method in a general approach to the controller design problem. As such, this paper illustrates the power and versatillity of the Second Method which might not be obvious to those only moderately familiar with the concept. On the other hand, a close study of this material indicates two drawbacks to the technique:

1) The most important drawback is the need for all of the plant outputs to generate the control signal. In higher-order systems these derivatives cannot be generated due to noise.
2) In any practical situation it would be very difficult to determine the rate of convergence to equilibrium in response to command signals.

References

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3. Grayson, L. P., "Design Via Liapunov's Second Method", Fourth Joint Automatic Control Conference A.I.Ch.E., June 1963.